Inverse eigenvalue problem for symmetric tridiagonal quadratic matrix polynomials

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In this paper we consider the finite element model corresponding to a real system where the mass, damping, and stiffness matrices are all symmetric tridiagonal. We show that the model can be constructed from five real eigenvalues and six real eigenvectors. We provide a necessary and sufficient condition for the existence of solution to this problem. Besides, we provide an analytical solution to this problem.

Keywords: *Quadratic matrix polynomial, inverse quadratic eigenvalue problem, symmetric tridiagonal matrix.*

1. Introduction

Consider a finite element model corresponding to a real system

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = f(t) \tag{1}$$

where *u(t)* is a vector of size *n, f (t)* is a time-dependent external force vector and $M, C, K \in \mathbb{R}^{n\times n}$. Such system arises in many important applications, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, signal processing, and finite element discretization of PDEs etc. In practical applications *M, C, K* are known as mass, damping and stiffness matrices respectively. By the separation of variables $u(t) = e^{\lambda t}x$, where *x* is a constant vector, we can get the general solution to the homogeneous equation of (1) and this solution is given in terms of the solution of the following *Quadratic Eigenvalue Problem* (QEP):

$$Q(\lambda)x := (\lambda^2 M + \lambda C + K)x = 0. \tag{2}$$

The scalar λ and the associated nonzero vector *x* are, respectively, called the eigenvalue and the eigenvector of the quadratic polynomial Q(λ). Indeed, the pair (λ, x) is known as eigenpair of Q(λ). Thus, λ is said to be an eigenvalue of Q(λ) if and only if $\det(\lambda^2 M + \lambda C + K) = 0$. Clearly, $Q(\lambda) = \lambda^2 M + \lambda C + K \in \mathbb{R}^{n\times n}[\lambda]$ has 2n finite eigenvalues if M is nonsingular. The quadratic eigenvalue problem is to find eigenvalues and eigenvectors of a quadratic matrix polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K \in \mathbb{R}^{n\times n}[\lambda]$. A good survey on the applications, mathematical properties and numerical methods of QEPs is included in [12] by Tisseur and Meerbergen.

On the contrary, *Inverse Quadratic Eigenvalue Problem* (IQEP) is to construct the matrices *M, C, K* of order $n \times n$ such that (λ_i, x_i), $i = 1, \ldots, p \leq 2n$ are eigenpairs of $Q(\lambda) = \lambda^2 M + \lambda C + K$. Very often, the matrices *M, C, K* are symmetric and tridiagonal and the tridiagonal structure comes from the inner connectivity of the elements in the original physical configuration. Therefore quadratic inverse eigenvalue problem should be solved with those structure constraints using partial eigen information. In that context it can be mentioned that the state of the art method is capable of computing only a few eigenpairs of Q(λ). In particular, we focus on the *Symmetric Tridiagonal Inverse Quadratic Eigenvalue Problem* (STIQEP). The problem is stated as follows.

Problem (P): Determine the real symmetric tridiagonal matrices *M, C, K* in such a manner that $(\lambda_i, x^i) \in \mathbb{R} \times \mathbb{R}^n$, $i = 1, \ldots, 5$ are eigenpairs of $Q(\lambda) := \lambda^2 M + \lambda C + K \in \mathbb{R}^{n\times n}[\lambda]$ and x^6 is a real eigenvector of Q(λ) provided that trace(M) = w is given.

IQEP has its practical applications in control design, antenna array processing, exploration and remote sensing, circuit theory, molecular spectroscopy, mechanical system simulation, structure analysis, particle physics and so on [4]. Recent developments include the finite element model updating problems in structural dynamics (see [5], [8]) and the partial eigenstructure assignment problems in control theory [6], [7]. Ram [9] studied the problem of reconstruction of undamped quadratic polynomial from two spectra whereas Ram [10] considered the same problem from a single eigenvalue, two eigenvectors. In [11], Ram and Elhay reconstructed a symmetric tridiagonal quadratic monic polynomial when two eigenvalues are given. In [2], Bai determined the matrices *C, K* with serially linked structure so that $Q(\lambda) = \lambda^2 I + \lambda C + K$ has a self conjugate set of four prescribed eigenpairs. However, in [1], Bai reconstructed the serially linked structured polynomial $Q(\lambda) = \lambda^2 M + \lambda C + K$from two real eignvalues and three real eigenvectors.

In this paper, we solve the Problem **(P)** and prescribed a necessary and sufficient condition for the solvability of Problem **(P)**. Indeed, we present an analytical solution of Problem **(P)**. Further, we proposed a necessary and sufficient condition for unique solution of Problem **(P)**.

Notation. Here $\mathbb{R}$ denotes the field of real numbers. We denote $\mathbb{R}^{n\times n}[\lambda]$ as the space of one parameter (λ) matrix polynomials whose coefficient matrices are of order n × n with its entries are from the field $\mathbb{R}$. Finally, I denotes the identity matrix of compatible size.

2. Solution of Problem (P)

At first we define the symmetric tridiagonal matrices as

$$M = \begin{bmatrix} a_1 & b_1 & & & & \\ b_1 & a_2 & b_2 & & & \\ & b_2 & a_3 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{n-2} & a_{n-1} & b_{n-1} \\ & & & & b_{n-1} & a_n \end{bmatrix}, \tag{3}$$

$$C = \begin{bmatrix} c_1 & d_1 & & & & \\ d_1 & c_2 & d_2 & & & \\ & d_2 & c_3 & d_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & d_{n-2} & c_{n-1} & d_{n-1} \\ & & & & d_{n-1} & c_n \end{bmatrix}, \tag{4}$$

$$\text{and} \qquad K = \begin{bmatrix} e_1 & f_1 & & & & \\ f_1 & e_2 & f_2 & & & \\ & f_2 & e_3 & f_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & f_{n-2} & e_{n-1} & f_{n-1} \\ & & & & f_{n-1} & e_n \end{bmatrix}. \tag{5}$$

Let λ_6 be an eigenvalue of $Q(\lambda) = \lambda^2 M + \lambda C + K$ corresponding to the known eigenvector x^6. Since (λ^i, x^i), $i = 1, \ldots, 6$ are eigenpairs of $Q(\lambda) = \lambda^2 M + \lambda C + K$, so the matrices *M, C, K* must satisfy

$$(\lambda_i^2 M + \lambda_i C + K)x^i = 0, \quad i = 1, \ldots, 6. \tag{6}$$

Thus, the equation (6) boils down to

$$\lambda_i^2 x_{j-1}^i b_{j-1} + \lambda_i^2 x_j^i a_j + \lambda_i^2 x_{j+1}^i b_j + \lambda_i x_{j-1}^i d_{j-1} + \lambda_i x_j^i c_j + \lambda_i x_{j+1}^i d_j + x_{j-1}^i f_{j-1} + x_j^i e_j + x_{j+1}^i f_j = 0 \tag{7}$$

for i = 1, . . . , 6, j = n – 1, n – 2, . . . , 1, and

$$\lambda_i^2 x_{n-1}^i b_{n-1} + \lambda_i^2 x_n^i a_n + \lambda_i x_{n-1}^i d_{n-1} + \lambda_i x_n^i c_n + x_{n-1}^i f_{n-1} + x_n^i e_n = 0 \tag{8}$$

for i = 1, . . . , 6, where we assumed that $x_0^i = 0, \ i = 1, \ldots, 6$. Without loss of generality assume that $a_n \neq 0$, then we set

$$\begin{cases} \widetilde{a}_j = a_j/a_n, & \widetilde{b}_j = b_j/a_n, \\ \widetilde{c}_j = c_j/a_n, & \widetilde{d}_j = d_j/a_n, \\ \widetilde{e}_j = e_j/a_n, & \widetilde{f}_j = f_j/a_n. \end{cases}$$

On dividing the equation (8) by a_n, we obtain

$$\lambda_i^2 x_{n-1}^i \widetilde{b}_{n-1} + \lambda_i x_n^i \widetilde{c}_n + \lambda_i x_{n-1}^i \widetilde{d}_{n-1} + x_n^i \widetilde{e}_n + x_{n-1}^i \widetilde{f}_{n-1} = -\lambda_i^2 x_n^i, \ i = 1, \ldots, 6. \tag{9}$$

Therefore, the scalars $\widetilde{b}_{n-1}, \widetilde{c}_n, \widetilde{d}_{n-1}, \widetilde{e}_n, \widetilde{f}_{n-1}$ must be determined in such a way that it satisfies the system of equation

$$\begin{bmatrix} \lambda_1^2 x_{n-1}^1 & \lambda_1 x_n^1 & \lambda_1 x_{n-1}^1 & x_n^1 & x_{n-1}^1 \\ \lambda_2^2 x_{n-1}^2 & \lambda_2 x_n^2 & \lambda_2 x_{n-1}^2 & x_n^2 & x_{n-1}^2 \\ \lambda_3^2 x_{n-1}^3 & \lambda_3 x_n^3 & \lambda_3 x_{n-1}^3 & x_n^3 & x_{n-1}^3 \\ \lambda_4^2 x_{n-1}^4 & \lambda_4 x_n^4 & \lambda_4 x_{n-1}^4 & x_n^4 & x_{n-1}^4 \\ \lambda_5^2 x_{n-1}^5 & \lambda_5 x_n^5 & \lambda_5 x_{n-1}^5 & x_n^5 & x_{n-1}^5 \\ \lambda_6^2 x_{n-1}^6 & \lambda_6 x_n^6 & \lambda_6 x_{n-1}^6 & x_n^6 & x_{n-1}^6 \end{bmatrix} \begin{bmatrix} \widetilde{b}_{n-1} \\ \widetilde{c}_n \\ \widetilde{d}_{n-1} \\ \widetilde{e}_n \\ \widetilde{f}_{n-1} \end{bmatrix} = - \begin{bmatrix} \lambda_1^2 x_n^1 \\ \lambda_2^2 x_n^2 \\ \lambda_3^2 x_n^3 \\ \lambda_4^2 x_n^4 \\ \lambda_5^2 x_n^5 \\ \lambda_6^2 x_n^6 \end{bmatrix}. \tag{10}$$

Since we are interested in nontrivial solution of (10) so the scalar λ_6 must be determined in such a way that

$$\det\left(\begin{bmatrix}\lambda_1^2x_n^1 & \lambda_1^2x_{n-1}^1 & \lambda_1x_n^1 & \lambda_1x_{n-1}^1 & x_n^1 & x_{n-1}^1\\ \lambda_2^2x_n^2 & \lambda_2^2x_{n-1}^2 & \lambda_2x_n^2 & \lambda_2x_{n-1}^2 & x_n^2 & x_{n-1}^2\\ \lambda_3^2x_n^3 & \lambda_3^2x_{n-1}^3 & \lambda_3x_n^3 & \lambda_3x_{n-1}^3 & x_n^3 & x_{n-1}^3\\ \lambda_4^2x_n^4 & \lambda_4^2x_{n-1}^4 & \lambda_4x_n^4 & \lambda_4x_{n-1}^4 & x_n^4 & x_{n-1}^4\\ \lambda_5^2x_n^5 & \lambda_5^2x_{n-1}^5 & \lambda_5x_n^5 & \lambda_5x_{n-1}^5 & x_n^5 & x_{n-1}^5\\ \lambda_6^2x_n^6 & \lambda_6^2x_{n-1}^6 & \lambda_6x_n^6 & \lambda_6x_{n-1}^6 & x_n^6 & x_{n-1}^6\end{bmatrix}\right)=0 \tag{11}$$

and

$$\text{rank}\left(\begin{bmatrix}\lambda_1^2x_{n-1}^1 & \lambda_1x_n^1 & \lambda_1x_{n-1}^1 & x_n^1 & x_{n-1}^1\\ \lambda_2^2x_{n-1}^2 & \lambda_2x_n^2 & \lambda_2x_{n-1}^2 & x_n^2 & x_{n-1}^2\\ \lambda_3^2x_{n-1}^3 & \lambda_3x_n^3 & \lambda_3x_{n-1}^3 & x_n^3 & x_{n-1}^3\\ \lambda_4^2x_{n-1}^4 & \lambda_4x_n^4 & \lambda_4x_{n-1}^4 & x_n^4 & x_{n-1}^4\\ \lambda_5^2x_{n-1}^5 & \lambda_5x_n^5 & \lambda_5x_{n-1}^5 & x_n^5 & x_{n-1}^5\\ \lambda_6^2x_{n-1}^6 & \lambda_6x_n^6 & \lambda_6x_{n-1}^6 & x_n^6 & x_{n-1}^6\end{bmatrix}\right)=\text{rank}\left(\begin{bmatrix}\lambda_1^2x_n^1 & \lambda_1^2x_{n-1}^1 & \lambda_1x_n^1 & \lambda_1x_{n-1}^1 & x_n^1 & x_{n-1}^1\\ \lambda_2^2x_n^2 & \lambda_2^2x_{n-1}^2 & \lambda_2x_n^2 & \lambda_2x_{n-1}^2 & x_n^2 & x_{n-1}^2\\ \lambda_3^2x_n^3 & \lambda_3^2x_{n-1}^3 & \lambda_3x_n^3 & \lambda_3x_{n-1}^3 & x_n^3 & x_{n-1}^3\\ \lambda_4^2x_n^4 & \lambda_4^2x_{n-1}^4 & \lambda_4x_n^4 & \lambda_4x_{n-1}^4 & x_n^4 & x_{n-1}^4\\ \lambda_5^2x_n^5 & \lambda_5^2x_{n-1}^5 & \lambda_5x_n^5 & \lambda_5x_{n-1}^5 & x_n^5 & x_{n-1}^5\\ \lambda_6^2x_n^6 & \lambda_6^2x_{n-1}^6 & \lambda_6x_n^6 & \lambda_6x_{n-1}^6 & x_n^6 & x_{n-1}^6\end{bmatrix}\right).$$

Further, dividing the equation (7) by a_n, it yields

$$\begin{bmatrix}\lambda_1^2x_j^1 & \lambda_1^2x_{j-1}^1 & \lambda_1x_j^1 & \lambda_1x_{j-1}^1 & x_j^1 & x_{j-1}^1\\ \lambda_2^2x_j^2 & \lambda_2^2x_{j-1}^2 & \lambda_2x_j^2 & \lambda_2x_{j-1}^2 & x_j^2 & x_{j-1}^2\\ \lambda_3^2x_j^3 & \lambda_3^2x_{j-1}^3 & \lambda_3x_j^3 & \lambda_3x_{j-1}^3 & x_j^3 & x_{j-1}^3\\ \lambda_4^2x_j^4 & \lambda_4^2x_{j-1}^4 & \lambda_4x_j^4 & \lambda_4x_{j-1}^4 & x_j^4 & x_{j-1}^4\\ \lambda_5^2x_j^5 & \lambda_5^2x_{j-1}^5 & \lambda_5x_j^5 & \lambda_5x_{j-1}^5 & x_j^5 & x_{j-1}^5\\ \lambda_6^2x_j^6 & \lambda_6^2x_{j-1}^6 & \lambda_6x_j^6 & \lambda_6x_{j-1}^6 & x_j^6 & x_{j-1}^6\end{bmatrix}\begin{bmatrix}\widetilde{a}_j\\ \widetilde{b}_{j-1}\\ \widetilde{c}_j\\ \widetilde{d}_{j-1}\\ \widetilde{e}_j\\ \widetilde{f}_{j-1}\end{bmatrix}=-\begin{bmatrix}\lambda_1^2x_{j+1}^1 & \lambda_1x_{j+1}^1 & x_{j+1}^1\\ \lambda_2^2x_{j+1}^2 & \lambda_2x_{j+1}^2 & x_{j+1}^2\\ \lambda_3^2x_{j+1}^3 & \lambda_3x_{j+1}^3 & x_{j+1}^3\\ \lambda_4^2x_{j+1}^4 & \lambda_4x_{j+1}^4 & x_{j+1}^4\\ \lambda_5^2x_{j+1}^5 & \lambda_5x_{j+1}^5 & x_{j+1}^5\\ \lambda_6^2x_{j+1}^6 & \lambda_6x_{j+1}^6 & x_{j+1}^6\end{bmatrix}\begin{bmatrix}\widetilde{b}_j\\ \widetilde{d}_j\\ \widetilde{f}_j\end{bmatrix}. \tag{12}$$

for $j = n-1, n-2, \dots, 1$.

Hence, to formulate our main result, we define

$$g^{(n)} := -\begin{bmatrix}\lambda_1^2x_n^1\\ \lambda_2^2x_n^2\\ \lambda_3^2x_n^3\\ \lambda_4^2x_n^4\\ \lambda_5^2x_n^5\\ \lambda_6^2x_n^6\end{bmatrix},\ \widetilde{w}^{(n)} := \begin{bmatrix}\widetilde{b}_{n-1}\\ \widetilde{c}_n\\ \widetilde{d}_{n-1}\\ \widetilde{e}_n\\ \widetilde{f}_{n-1}\end{bmatrix},\ A_{nn} := \begin{bmatrix}\lambda_1^2x_{n-1}^1 & \lambda_1x_n^1 & \lambda_1x_{n-1}^1 & x_n^1 & x_{n-1}^1\\ \lambda_2^2x_{n-1}^2 & \lambda_2x_n^2 & \lambda_2x_{n-1}^2 & x_n^2 & x_{n-1}^2\\ \lambda_3^2x_{n-1}^3 & \lambda_3x_n^3 & \lambda_3x_{n-1}^3 & x_n^3 & x_{n-1}^3\\ \lambda_4^2x_{n-1}^4 & \lambda_4x_n^4 & \lambda_4x_{n-1}^4 & x_n^4 & x_{n-1}^4\\ \lambda_5^2x_{n-1}^5 & \lambda_5x_n^5 & \lambda_5x_{n-1}^5 & x_n^5 & x_{n-1}^5\\ \lambda_6^2x_{n-1}^6 & \lambda_6x_n^6 & \lambda_6x_{n-1}^6 & x_n^6 & x_{n-1}^6\end{bmatrix},$$

and for $j = n-1, \dots, 1$ we set

$$A_{jj} := \begin{bmatrix}\lambda_1^2x_j^1 & \lambda_1^2x_{j-1}^1 & \lambda_1x_j^1 & \lambda_1x_{j-1}^1 & x_j^1 & x_{j-1}^1\\ \lambda_2^2x_j^2 & \lambda_2^2x_{j-1}^2 & \lambda_2x_j^2 & \lambda_2x_{j-1}^2 & x_j^2 & x_{j-1}^2\\ \lambda_3^2x_j^3 & \lambda_3^2x_{j-1}^3 & \lambda_3x_j^3 & \lambda_3x_{j-1}^3 & x_j^3 & x_{j-1}^3\\ \lambda_4^2x_j^4 & \lambda_4^2x_{j-1}^4 & \lambda_4x_j^4 & \lambda_4x_{j-1}^4 & x_j^4 & x_{j-1}^4\\ \lambda_5^2x_j^5 & \lambda_5^2x_{j-1}^5 & \lambda_5x_j^5 & \lambda_5x_{j-1}^5 & x_j^5 & x_{j-1}^5\\ \lambda_6^2x_j^6 & \lambda_6^2x_{j-1}^6 & \lambda_6x_j^6 & \lambda_6x_{j-1}^6 & x_j^6 & x_{j-1}^6\end{bmatrix},\quad \widetilde{w}^{(j)} := \begin{bmatrix}\widetilde{a}_j\\ \widetilde{b}_{j-1}\\ \widetilde{c}_j\\ \widetilde{d}_{j-1}\\ \widetilde{e}_j\\ \widetilde{f}_{j-1}\end{bmatrix},$$

Based on the above analysis, we prescribe a necessary and sufficient condition for the solvability of Problem **(P)**, which is as follows.

Theorem 2.1. *Problem (P) has a nontrivial solution if and only if the following conditions are satisfied:*

i. The scalar λ_6 is determined by solving the equation (11) *in such a way that* $\operatorname{rank}(A_{nn}) = \operatorname{rank}\left(\begin{bmatrix} A_{nn} & g^{(n)} \end{bmatrix}\right)$,

ii. $\operatorname{rank}(A_{jj}) = \operatorname{rank}\left(\begin{bmatrix} A_{jj} & B_{jj}\widetilde{z}^{j} \end{bmatrix}\right)$, *for all* $j = n-1, \ldots, 1$.

Proof: Clearly, the Problem **(P)** has a nontrivial solution if and only if the equations (10) and (12) has nontrivial solutions for each $j = n - 1, \ldots, 1$.

Corollary 2.2. Problem ***(P)*** has an unique nontrivial solution if and only if the following conditions are met:

i. $\operatorname{rank}(A_{nn}) = \operatorname{rank}\left(\begin{bmatrix} A_{nn} & g^{(n)} \end{bmatrix}\right) = 5$,

ii. $\det(A_{jj}) \neq 0$ *for all* $j = n-1, \ldots, 1$.

Our next remark describes a procedure to construct the symmetric tridiagonal matrices M, C, K in equations (3), (4) and (5) respectively.

Remark 2.3. If the conditions of Theorem 2.1 are met then we obtain the real numbers $\widetilde{b}_{n-1},\ \widetilde{c}_n,\ \widetilde{d}_{n-1},\ \widetilde{e}_n,\ \widetilde{f}_{n-1}$ by solving the system of equation (10) and thereafter solving the equation (12) successively for j = n – 1,...,1 we obtain the real numbers $\widetilde{a}_j,\ \widetilde{b}_{j-1},\ \widetilde{c}_j,\ \widetilde{d}_{j-1},\ \widetilde{e}_j,\ \widetilde{f}_{j-1}$. Compute $\widetilde{w} = 1 + \Sigma_{j=1}^{n-1}\widetilde{a}_j$. Finally, the real triples $(a_j, c_j, e_j)_{j=1}^{n},\ (b_j, d_j, f_j)_{j=1}^{n-1}$ are obtained from

$$\begin{cases} a_n = w/\widetilde{w}, \quad a_j = \widetilde{a}_j w/\widetilde{w}, \quad j = 1, \ldots, n-1, \\ b_j = \widetilde{b}_j w/\widetilde{w}, \quad j = 1, \ldots, n, \\ c_j = \widetilde{c}_j w/\widetilde{w}, \quad d_j = \widetilde{d}_j w/\widetilde{w}, \quad j = 1, \ldots, n, \\ e_j = \widetilde{e}_j w/\widetilde{w}, \quad f_j = \widetilde{f}_j w/\widetilde{w}, \quad j = 1, \ldots, n \end{cases} \tag{13}$$

where $w = \operatorname{trace}(M) = \Sigma_{j=1}^{n} a_j$ *is given by Problem **(P)**. Then we construct the matrices M, C, K in equations* (3), (4), (5) *by plugging the values of* $(a_j, c_j, e_j)_{j=1}^{n},\ (b_j, d_j, f_j)_{j=1}^{n-1}$ from (13).

Next we state the result from [3] for the solution of a system of linear equations in (10) and (12).

Lemma 2.4. *A system of linear equations Ax = b has a solution if and only if* $AA^{\dagger}b = b$ *and all of its solutions are given by*

$$x = A^{\dagger}b + (I - A^{\dagger}A)z \tag{14}$$

where $A^{\dagger}$ *denotes the Moore-Penrose pseudoinverse of A and z is an arbitrary vector of compatible size.*

Proof: For proof we refer the readers to [3].

3. Conclusion

In this paper, we have dealt with reconstructing a symmetric tridiagonal quadratic matrix polynomial whenever its five real eigenvalues and six real eigenvectors are given. We present a necessary and sufficient condition for existence of solution to this problem. Also, we obtained an explicit computable expression of the coefficient matrices of the quadratic matrix polynomial. The problem discussed in this paper is applicable only for real eigenpairs, but it can be extended for complex eigenpairs.

References

1. Z.J. Bai, *Constructing the physical parameters of a damped vibrating system from eigendata,* Linear Algebra and its Applications, 428.2-3 (2008) pp.625-656.
2. Z.J. Bai, *Symmetric tridiagonal inverse quadratic eigenvalue problems with partial eigendata,* Inverse problems 24.1 (2007) 015005.
3. A. Ben-Israel, and T.N.E. Greville, *Generalized inverses: theory and applications*, Vol. 15. Springer Science & Business Media, 2003.
4. M.T. Chu, *Inverse eigenvalue problems,* SIAM review. 40(1), 1998, pp.1-39.
5. M.T. Chu, Y.C. Kuo, and W.W. Lin, *On inverse quadratic eigenvalue problems with partially prescribed eigenstructure*, SIAM J. Matrix Anal. Appl., 25 (2004), pp.995-1020.
6. M. Chu, N.D. Buono, and B. Yu, *Structured quadratic inverse eigenvalue problem, I. Serially linked systems,* SIAM J. Sci. Comp. 29.6 (2007) pp.2668-2685.
7. B.N. Datta, S. Elhay, Y.M. Ram, and D.R. Sarkissian, *Partial eigenstructure assignment for the quadratic pencil*, J. Sound Vib. 230 (2000), pp.101-110.
8. M.I. Friswell and J.E. Mottershead, *Finite Element Model Updating in Structural Dynamics,* Kluwer Academic Publishers, 1995.
9. Y.M. Ram, *Inverse eigenvalue problems for a modified vibrating system*, SIAM J. Appl. Math. 53 (1993), pp.1175-1762.
10. Y.M. Ram, *Inverse mode problems for the discrete model of a vibrating beam*, J. Sound Vib. 169 (1994) pp.239-252.
11. Y.M. Ram, and S. Elhay, *An inverse eigenvalue problem for the symmetric tridiagonal quadratic pencil with application to damped oscillatory systems,* SIAM J. Appl. Math. 56 (1996) pp.232-244.
12. F. Tisseur, and K. Meerbergen, *The quadratic eigenvalue problem,* SIAM review, 43(2001), pp.235-286.